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The Propagation of Weak Shocks in a Stratified Atmosphere

G. B. WHITHAM

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THE PROPAGATION OF WEAK SHOCKS IN A STRATIFIED ATMOSPHERE

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The propagation of weak shocks in a stratified atmosphere

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G. B. Whitham

Introduction

In a previous report [1], a complete theory was presented of the propagation of weak spherically symmetric shocks in air of uniform density, pressure etc: Later, the theory was extended [2] to a case in which the undisturbed fluid into which the shock moved was not uniform, but the property of spherical symmetry was preserved. In the present report, a combination of the methods of [1] and [2] is applied to the problem of a shock (produced by an explosion say) propagating in the atmosphere, taking into account the variation of the density, pressure etc. of the air with height above the ground. Since the air is stratified and, in the most important problem, the shock is initially spherical (approximately) only the symmetry about a vertical axis remains; the physical quantities are now functions of <u>two</u> space variables and the time. The introduction of an additional independent variable into the problem is far from a trivial extension, and if the theory can be worked out successfully in this particular case, it will serve as a prototype for many other investigations (in the theory of three dimensional steady supersonic flow, for example). The present account constitutes a "progress report" since certain points require further consideration and justification before a final report can be written.

Only weak disturbances are treated here, and as before the linearized (acoustic) theory forms the starting point of the method. The linear results are inadequate in certain ways, as explained in [1], but their shortcomings can be remedied to give a valid description of the flow. The work, therefore, falls into two sections: (I) the derivation and consideration of the appropriate results of linear theory, and (II) the improvement of those results and the determination of the shock. The first section, strangely enough, is the more troublesome and the more incomplete, since explicit solutions of the linear equations are not known in general, and we have to be satisfied with a certain expansion solution. This expansion ,

however, is of considerable interest for its own sake and is of general application to linear hyperbolic equations of second order. The first section extends the methods of treating the linear theory, which were developed in [2], to include an additional space variable (the treatment of linear theory is trivial in the problem of [1], since the explicit solution of spherical sound waves is well-known), while the second section extends the applicability of the basic ideas described in [1].

I. The linearized theory

The physical quantities are functions of the distance, r, from the (vertical) axis of symmetry, the height, z, above the ground, and the time t. If u and w are the velocity components in the directions of the r and z axes, respectively, the density is ρ and the pressure is p, then the equations of momentum (including the acceleration of gravity g) are

$$(1)\qquad u_t + uu_r + wu_z = -\frac{1}{\rho}\, p_r$$

$$(2)\qquad w_t + uw_r + ww_z = -\frac{1}{\rho}\, p_z - g,$$

and the equation of continuity is

$$(3)\qquad \rho_t + u\rho_r + w\rho_z + \rho(u_r + \frac{u}{r} + w_z) = 0.$$

It may be assumed, since entropy changes are negligible for weak disturbances, that p is a given function of ρ. Then, introducing the velocity potential ϕ, equations (1) and (2) may be integrated to give Bernoulli's equation

$$(4)\qquad \phi_t + \tfrac{1}{2}(\phi_r^2 + \phi_z^2) + \int\frac{dp}{\rho} + gz = \text{constant}.$$

In equilibrium $u = w = 0$, and $p = P(z)$, $\rho = R(z)$, $a = \sqrt{dp/d\rho} = A(z)$, say,

and from (2) this equilibrium configuration is determined by

$$(5)\qquad P'(z) + gR(z) = 0.$$

To obtain the linearized equation of motion, the deviations of ϕ, p, ρ, amd a from their equilibrium values (the undisturbed value of ϕ is zero) are assumed to be small, and in the above equations only the first order terms in these small quantities are retained.

Equation (4) becomes

(6) $$p - P(z) = -R(z)\,\phi_t,$$

and (3) becomes

(7) $$\frac{\rho_t}{R} + \phi_{rr} + \frac{\phi_r}{r} + \frac{1}{R}\frac{\partial}{\partial z}(R\phi_z) = 0.$$

In addition, since $a^2 = dp/d\rho$, $\rho_t \doteq (p-P)_t/A^2(z)$; hence, from (6), (7) is

(8) $$\phi_{tt} = A^2(z)\left\{\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} + \frac{R'(z)}{R(z)}\phi_z\right\}.$$

The solution of this equation for ϕ is not known, but an expansion valid near the front of the disturbance will prove to be of value. To motivate this expansion we consider two special cases: (i) the case of a uniform atmosphere with A equal to a constant A_o and ϕ a function of r alone (this is the problem of cylindrical sound waves), and (ii) the case of plane waves propagating vertically so that ϕ is a function only of z. In each of these problems only one space variable appears, and the equation reduces to the type considered in [2]; (i) and (ii) are now considered in turn.

(i) Cylindrical sound waves

In this case (8) reduces to

(9) $$\phi_{rr} + \frac{\phi_r}{r} = \frac{\phi_{tt}}{A_o^2},$$

and it is well-known that the solution which represents an outgoing wave starting at $t = 0$ is

(10) $$\phi = K_o(pr/A_o)\,h(t) = \int_0^{t-r/A_o} \frac{h(t')\,dt'}{\sqrt{(t-t')^2 - r^2/A_o^2}}$$

where $h(t)$ is an arbitrary function, K_o is the Bessel function, and p denotes the operator in the Heaviside calculus. (The pressure has also been denoted by p, but no confusion will ensue, since the pressure does not appear again in this section).

It is found in these problems (see [1] and [2]) that in order to determine the head shock the linearized solution is required in the region where $A\xi/r$ is small; here, ξ is the characteristic vari-

able $t-r/A_0$ which measures the time after the arrival of the first disturbance (which travels on $\xi = 0$). Although small ξ corresponds to the front of the disturbance in the linearized theory, the solution for small ξ is not sufficient. For in the improved (non-linear) theory, characteristics are continually meeting the shock so that eventually at large distances the appropriate value of the characteristic variable near the shock is no longer small. Hence, roughly speaking, we require the solution for small ξ and, in addition, the solution for large r, more precisely, the solutions for small $A\xi/r$ is needed.

The expansion can be obtained from the integral in (10), but is found more quickly from the Heaviside representation of the solution. Small values of ξ correspond to large values of p in the Heaviside representation; in fact, 1/p plays a role corresponding to ξ so that the expansion of ϕ for small values of $A_0\xi/r$ is found from the Heaviside representation by assuming A_0/pr is small. Expanding the Bessel function for large values of its argument, pr/A_0, and interpreting term by term, we have

$$(11)\qquad \phi \sim \sqrt{\frac{\pi A_0}{2pr}}\; e^{-pr/A_0} \left\{1 - \frac{1}{8}\frac{A_0}{pr} + \frac{9}{128}\frac{A_0^2}{p^2r^2} - \dots\right\} h(t)$$

$$= \left(\frac{A_0}{r}\right)^{\frac{1}{2}} f(t-r/A_0) - \frac{1}{8}\left(\frac{A_0}{r}\right)^{\frac{3}{2}} f_1(t-r/A_0) + \frac{9}{128}\left(\frac{A_0}{r}\right)^{\frac{5}{2}} f_2(t-r/A_0) - \dots,$$

where

$$f(t) = \sqrt{\frac{\pi}{2p}}\, h(t) = \frac{1}{\sqrt{2}} \int_0^t \frac{h(t')\,dt'}{\sqrt{t-t'}},$$

and $f_n(t)$ denotes the nth repeated integral from 0 to t. To a first order approximation only the first term in (11) need be retained; using it, the linearized theory may be improved and the head shock determined by the procedure described in [1]. The details are not given here, since this special case is only used to motivate the later treatment of the linear equation (8); in any case, the results are very similar to those of [3].

(ii) Vertical propagation of plane waves

It is assumed now that the air is polytropic i.e. the pressure is equal to $k\rho^\gamma$, where k and γ are constants. It follows then from

(5) that $R(z) \propto (z_o - z)^{\frac{1}{\gamma-1}}$, and therefore that

(12) $$A = A_1(z_o - z)^{\frac{1}{2}},$$

where z_o and A_1 are constants. As $z \to z_o$, R and A tend to zero, but of course the simple polytropic ceases to apply to the atmosphere before $z = z_o$ is reached (for example, the tropopause, at which $T \propto A^2$ is discontinuous, intervenes); however, the behaviour according to (12) of A near $z = z_o$ will prove interesting mathematically. With these values of R and A, and ϕ a function of z alone, (8) reduces to

(13) $$\phi_{tt} = A_1^2(z_o - z)\left\{\phi_{zz} - \frac{1}{\gamma-1}\frac{\phi z}{z_o - z}\right\}.$$

It is convenient to introduce

(14) $$\eta(z) = \int_z^{z_o} \frac{dz}{A(z)} = 2A_1^{-1}(z_o - z)^{\frac{1}{2}}$$

in place of z; then

(15) $$\phi_{tt} = \phi_{\eta\eta} - \frac{2\nu\phi_\eta}{\eta},$$

where $\nu = \frac{1}{2}(3-\gamma)/(\gamma-1)$, and the characteristics are $t \pm \eta$ = constant. Replacing ϕ_{tt} by the operational form $p^2\phi$, the general solution of (15) is

(16) $$\phi = \eta^{\nu+\frac{1}{2}}\left\{I_{\nu+\frac{1}{2}}(\eta p)h_1(t) + K_{\nu+\frac{1}{2}}(\eta p)h_2(t)\right\},$$

where $I_{\nu+\frac{1}{2}}$ and $K_{\nu+\frac{1}{2}}$ denote Bessel functions of imaginary argument, and $h_1(t), h_2(t)$ arbitrary functions. For simplicity, consider the special case ν = integer for which

$$I_{\nu+\frac{1}{2}}(\zeta) = \sqrt{\frac{2\zeta}{\pi}}\,\zeta^\nu\left(\frac{d}{\zeta d\zeta}\right)^\nu\left\{\frac{e^\zeta - e^{-\zeta}}{2\zeta}\right\}, K_{\nu+\frac{1}{2}}(\zeta) = (-1)^\nu\zeta^\nu\sqrt{\frac{\pi\zeta}{2}}\left(\frac{d}{\zeta d\zeta}\right)^\nu\left(\frac{e^{-\zeta}}{\zeta}\right).$$

Then, it is easy to pick out the solution representing propagation on the 'outgoing' characteristics $t + \eta$ = constant (note: with the definition (14), η decreases as z increases, hence the outgoing wave has the positive sign in the characteristic equation); it is

(17) $$\phi = \eta^\nu L(\eta p)j(t),$$

where $L(\zeta) = \zeta^{\nu+1}\left(\frac{d}{\zeta d\zeta}\right)^\nu\left(\frac{e^\zeta}{\zeta}\right)$, and j(t) is an arbitrary function.

Interpreting (17), we have the solution

(18) $$\phi = \eta^\nu j(t+\eta) - \frac{\nu(\nu+1)}{2}\eta^{\nu-1}j_1(t+\eta) + \ldots + \frac{(-1)^\nu(2\nu-1)!}{2^{\nu-1}(\nu-1)!}j_\nu(t+\eta),$$

where $j_n(t)$ denotes the nth integral of $j(t)$, the lower limits being the value of $t + \eta$ at the head of the wave. Near the head of the wave, the successive terms decrease in magnitude provided $\eta \neq 0$, and the solution is of exactly the same general form as (11); that is,

$$\phi \sim \phi^{(o)} f(\xi) + \phi^{(1)} f_1(\xi) + \dots, \tag{19}$$

where ξ is the characteristic variable defined to be zero at the head of the wave, $f(\xi)$ is an arbitrary function ($j(t+\eta) = f(\xi)$), and the ϕ's are certain functions of the space variable. In the particular case (18), there are only a finite number of terms, but for non-integral values of ν the series would be infinite. We note, however, that in this case the series is valid only for small ξ and $\eta \neq 0$; as $\eta \to 0$, it is not uniformly valid. For fixed ξ, the last term of (18) dominates for sufficiently small values of η; the order of the terms should be reversed, and the expression (18) (which in general is replaced by an infinite series) would involve successive derivatives of $j_\nu(t)$. Thus, we see that although (19) is valid in the initial stages of the propagation, it may not apply in later stages.

(iii) Discussion and generalization of the results of (i) and (ii)

An expansion of the form (19) can be found quite generally for an equation of the form

$$\phi_{tt} = A^2(z)\left\{\phi_{zz} + B(z)\,\phi_z + C(z)\phi\right\}, \tag{20}$$

even though the explicit solution is not known, by substituting (19) in the equation and equating the coefficients of the f's to zero. This was used extensively in [2]; in all cases (apart from the occurrence of singularities as in the problem discussed above), (19) is valid for small ξ, and in certain cases such as (11) its validity extends into the region $A\xi/r$ small. The expansion is intimately connected with the W.K.B. method of finding asymptotic solutions of ordinary differential equations. For, the operational form of (20) is

$$\phi_{zz} + B(z)\phi_z + \left\{C(z) - p^2/A^2(z)\right\}\phi = 0,$$

and the asymptotic solution for large p (corresponding to small ξ) is obtained in the W.K.B. method by the substitution

$$\phi \sim e^{-pq(z)} \left\{ \phi^{(o)}(z) + \frac{1}{p}\phi^{(1)}(z) + \dots \right\} \overline{f}(p),$$

where $\overline{f}(p)$ is an arbitrary function of p. It is found that $q(z) = \int A^{-1} dz$, hence, letting $\overline{f}(p)$ be the representation of $f(t)$, the expansion is exactly (19).

Another interesting feature of (19) is in the significance of $\phi^{(o)}$. Suppose, there is a discontinuity in $f(\xi)$, then, since $f_n(\xi)$ is continuous for $n \geq 1$, (19) shows that an initial discontinuity would be propagated along a characteristic and its magnitude is proportional to $\phi^{(o)}$. (Since ϕ is the velocity potential, discontinuities of ϕ itself would not arise. However, ϕ_t is proportional to the pressure increase, and $\phi_t \sim \phi^{(o)} f'(\xi) + \phi^{(1)} f(\xi) + \dots$; therefore, the same result is obtained for pressure jumps by considering a discontinuity in $f'(\xi)$). On the other hand, the propagation of such discontinuities can be deduced by certain standard methods directly from the coefficients of equation (20); if this is done, exactly the same function $\phi^{(o)}$ is found for the magnitude of the discontinuity as it moves along the characteristic. Thus (19) already contains the correct (linearized) propagation of discontinuities. Moreover, since we have a complete expansion near the head of the wave, additional information is furnished about the variation in the magnitude of the higher derivatives. For example, suppose as a special case of (19) we have

$$\phi \sim \phi^{(o)}(z)\,\xi + \tfrac{1}{2}\phi^{(1)}(z)\,\xi^2 + \dots .$$

Then, the discontinuities in ϕ_t and ϕ_z on $\xi = 0$ are proportional to $\phi^{(o)}$ and $-A^{-1}\phi^{(1)}$, the discontinuities in ϕ_{tt}, ϕ_{tz} and ϕ_{zz} on $\xi = 0$ are proportional to $\phi^{(1)}$, $d\phi^{(o)}/dz - A^{-1}\phi'^{(1)}$, and $\left\{A^{-2}\phi^{(1)} - 2A^{-1}d\phi^{(o)}/dz + A^{-2}\phi^{(o)}dA/dz\right\}$, respectively and so on.

Independently, M. Kline has considered a special case of (19) (and of the extensions which will be described in (iv)); he considers periodic waves and thus $f(\xi) = e^{iw\xi}$. Although it is convenient in our work to take the lower limits of the integrals in (19) as $\xi = 0$, the expansion is a solution of the equation whatever the limits are, or even if the integrals are indefinite; hence, $f_n(\xi)$ may be taken as $e^{iw\xi}/w^n$. In this way Kline has an expansion valid for large w (for any ξ).

(iv) The general problem

The extension of (19) to include an extra space variable is now discussed and in a similar way, any number of space variables could be treated. When an additional space variable is introduced, there no longer exists a unique set of characteristic surfaces $\xi(t,r,z)$ = constant, since the characteristic conoids from any curve in the plane $t = 0$ have an envelope which is a characteristic surface. However, the wave front $t = a(r,z)$ is uniquely defined, and the obvious extension of the ξ used in (19) is $\xi \equiv t - a(r,z)$, so that at any point in the (r,z) plane, ξ measures the time which has elapsed since the passage of the wave front. In addition, the ϕ's of (19) now become functions of r and z. As before, this expansion will be valid near the head of the wave since $f_n/f_{n-1} = O(\xi)$, and in certain cases the region of validity may be extended to include points at large distances, as in (i). When the polytropic assumption is made a behaviour similar to (ii) is expected; in fact, near $A=0$, any wave will tend, by refraction, to become parallel to the line $A=0$; hence the similarity to (ii) should be very close.

By substitution of the extension of (19) into the equation (8), we have

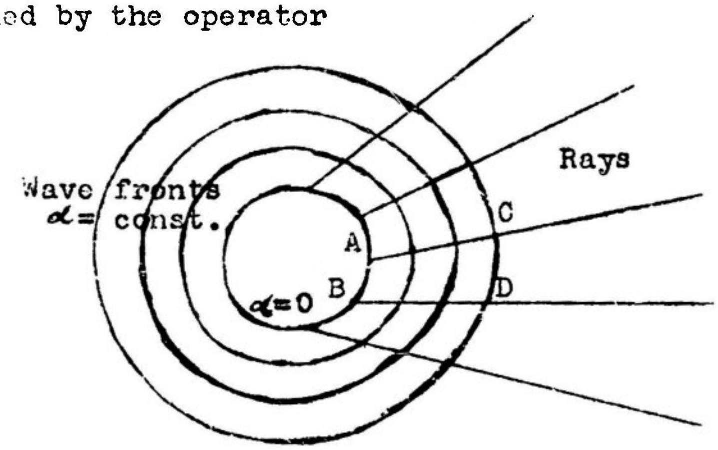
$$a_r^2 + a_z^2 = 1/R^2(z), \tag{21}$$

$$2a_r\,\phi_r^{(o)} + 2a_z\,\phi_z^{(o)} + \left\{a_{rr} + \frac{a_r}{r} + a_{zz} + \frac{R'(z)}{R(z)}a_z\right\}\phi^{(o)} = 0, \tag{22}$$

$$2a_r\,\phi_r^{(n)} + 2a_z\,\phi_z^{(n)} + \left\{a_{rr} + \frac{a_r}{r} + a_{zz} + \frac{R'(z)}{R(z)}a_z\right\}\phi^{(n)} = \phi_{rr}^{(n-1)} + \frac{\phi_r^{(n-1)}}{r} + \phi_{zz}^{(n-1)} + \frac{R'}{R}\phi_z^{(n-1)}, \text{ for } n \geq 1. \tag{23}$$

The first equation is the well-known eiconal equation which must be satisfied by the wave front. If, for example, the initial wave front at $t = 0$ is known i.e. the solution of $a(r,z) = 0$ is given, (21) is sufficient to determine the function $a(r,z)$. Geometrically, the surface $t = a(r,z)$ in the (t,r,z) space is the envelope of the characteristic conoids drawn from points on the initial wave front; the successive wave fronts in the (r,z) plane for different times are the projections (on the (r,z) plane) of the curves of intersection of the surface $t = a(r,z)$ with planes t constant.

It is convenient, in addition to the set of wave fronts α = constant, to consider the 'rays' which are the orthogonal trajectories of the wave fronts (Figure 1). The direction cosines of a ray at any point are $(A\alpha_r, A\alpha_z)$ (by (21) the sum of the squares is 1), and therefore the rate of change of any quantity along a ray is determined by the operator



$$\frac{d}{ds} = A\alpha_r \frac{\partial}{\partial r} + A\alpha_z \frac{\partial}{\partial z} \quad . \tag{24}$$

In particular, $d\alpha/ds = A(\alpha_r^2+\alpha_z^2) = 1/A$, therefore the wave front expands out along the rays with velocity A. In terms of the ray derivative the left hand sides of (22) and (23) become $d\phi^{(n)}/ds + K\phi^{(n)}$, where K is a known function of r and z, assuming that $\alpha(r,z)$ has already been determined. The equations for the ϕ's thus become simply first order ordinary differential equations along the rays.

Again, the propagation of discontinuities in the solution of equation (8) can be studied directly by an alternative method and exactly equations (21) and (22) are obtained for the wave fronts and the magnitude of the discontinuity. For this reason the treatment of these equations - the theory of geometrical optics and acoustics (see [4]) - is already known; the relevant part is repeated here. Equation (22) can be written

$$\frac{\partial}{\partial r}(\alpha_r \phi^{(o)^2} rR) + \frac{\partial}{\partial z}(\alpha_z \phi^{(o)^2} rR) = 0;$$

integrating this expression over the region ABCD (Fig. 1) contained

between any two wave fronts and two adjacent rays, the divergence theorem gives

(25) $$\int (\ell a_r + m a_z)\phi^{(o)2}\, rR\, d\sigma = 0,$$

where the integration is over the boundary of ABCD, and ℓ, m are the direction cosines of the outward normal. Since (Aa_r, Aa_z) are the direction cosines of the rays, $\ell a_r + m a_z = 0$ on AC and BD, while on AB and CD, $\ell a_r + m a_z = A(a_r^2 + a_z^2) = 1/A$. Therefore, (25) gives approximately

$$[\phi^{(o)2}\, rR\Delta\sigma/A]_1 - [\phi^{(o)2}\, rR\Delta\sigma/A]_o = 0,$$

where $\Delta\sigma$ denotes the length of the small segment of wave front cut off by the rays, and subscripts o and 1 indicate that the quantity is evaluated at AB and CD respectively. Defining the expansion ratio $E = \lim\limits_{\Delta\sigma_o \to o} \Delta\sigma_1/\Delta\sigma_o$, we have $\phi^{(o)2}\, rRE/A$ is constant on each ray, and in particular equals its value at the initial wave front. In this way, we may consider $a(r,z)$ and $\phi^{(o)}(r,z)$ to be determined.

Equations (23) for the later functions $\phi^{(n)}$, even though they are reduced by (24) to equations along the rays are complicated to deal with. An investigation of them would be of interest in yielding additional information about the region of validity of (19); also these later terms show, in a similar way to that described in (iii), how discontinuities in higher derivatives vary, according to the linear theory.

II <u>Improvements of the linearized theory and determination of the head shock.</u>

According to the linearized theory, in the early stages of the propagation and even later in certain cases, we have a solution of the form (19), and as a first approximation we take only the first term

(26) $$\phi = X(r,z)f(t-a(r,z)),$$

where for convenience in writing $\phi^{(o)}$ has been replaced by X. It is now shown how (26) may be improved and the head shock determined. The method will apply wherever ϕ is of the approximate form shown in (26) even if it no longer comes from an expansion (19). For

example, in the problem described in (ii), (19) did not apply near $\eta = 0$, but, near $r_\zeta = 0$, another expansion* whose dominant term was still of the form (26) (with different X and f), was valid; therefore, the method described below could still be applied in that region.

The functions $a(r,z)$ and $X(r,z)$ are assumed to be known from the geometrical acoustics described in the previous section and f is an arbitrary function which is fixed by applying some appropriate boundary or initial condition. Unfortunately, appropriate boundary conditions would be applied in regions outside the range of validity of the expansion (19). For example, if the acoustic waves in I(i) are produced by the small motion of a "cylindrical piston" of initial radius equal to zero, the boundary condition is applied where r is small, and $\phi \sim A_0 r^{-1} \int_0^t h(t')dt'$ there). However, in explosion problems, the general form of f is known from examination of special cases such as the problem in [1]; therefore, a suitable form can be assumed when required.

Henceforth, we shall be more interested in the values of u,w, and a than in the velocity potential ϕ; these are now set down. The dominant terms in u and w are

(27) $$u = a_r XF(t-a) ,$$

(28) $$w = a_z XF(t-a) ,$$

where $F(\xi) = -f'(\xi)$. The sound speed a is obtained from Bernoulli's equation (4), but it involves the pressure-density relation; for simplicity, we assume that the air is polytropic, although the general case could be carried out in the same way. Then

(29) $$a \doteq A - \frac{\gamma-1}{2A}\phi_t = A + \frac{\gamma-1}{2A} XF(t-a).$$

In accordance with the hypothesis which has been developed and discussed in detail in previous papers ([1], [2], [3]), the linearized theory, which becomes inadequate after the disturbance has propagated for some time, may be corrected quite simply to provide a

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* For integral values of ν, the two 'expansions' were the same expression written in different order; from the present point of view, however, they may be treated as distinct, since they would be in general.

valid description of the motion. In the exact non-linear theory, small 'wavelets' (whose paths are characteristic surfaces in the (t,r,z) space) move at the local sound speed a relative to the fluid velocity (u,w). The linear theory, however, approximates this value by the undisturbed sound speed, A, on the grounds that $a-A$, u, w are small compared to A. But, as a wavelet travels the error introduced by this assumption becomes large, since the small deviations in the propagation velocity accumulate. To correct this, the linearized characteristic variable $t-\alpha(r,z)$ must be replaced in (26)-(29) by $\tau(t,r,z)$, where τ is a sufficiently accurate approximation to the exact characteristic variable. Thus (27), (28), (29) become

(30) $$u = \alpha_r XF(\tau),$$

(31) $$w = \alpha_z XF(\tau),$$

(32) $$a = A + \frac{\gamma-1}{2A} XF(\tau),$$

where $\tau(t,r,z)$ remains to be determined.

A surface $\tau(t,r,z)$ = constant in the (t,r,z) space corresponds to the propagation of a wavelet, and gives a set of curves in the (r,z) plane, each curve corresponding to a single value of t and representing the position of the wavelet at that time. It is easy to show that the velocity of the wavelet normal to itself in the (r,z) plane is $-\tau_t/\sqrt{\tau_r^2+\tau_z^2}$. The velocity of the particles of fluid normal to the wavelet is $(u\tau_r + w\tau_z)/\sqrt{\tau_r^2+\tau_z^2}$; therefore the characteristic condition, that the velocity of the wavelet differs from this particle velocity by the sound speed, gives

$$\frac{-\tau_t}{\sqrt{\tau_r^2+\tau_z^2}} = \frac{u\tau_r + w\tau_z}{\sqrt{\tau_r^2+\tau_z^2}} \pm a,$$

or

(33) $$(\tau_t + u\tau_r + w\tau_z)^2 = a^2(\tau_r^2 + \tau_z^2).$$

Since expressions for u,w and a in terms of τ are known, (33) gives an equation to determine τ. Now, although τ differs from $t-\alpha$ by an amount which becomes large (this being one of the crucial points in

the discussion), the difference is still small compared to α. Hence, we may set

$$\tau = t - \alpha(r,z) + \mu(r,z),$$

where μ/α is small. Then, substituting in (33) and neglecting terms of second order in small quantities, we obtain

$$\alpha_r\mu_r + \alpha_z\mu_z = \frac{u\alpha_r + w\alpha_z}{A^2} + \frac{a-A}{A}, \tag{34}$$

(using (21)). From (30), (31) and (32), the right hand side of (34) is $\frac{1}{2}(\gamma+1)XF(\tau)/A^4$. Hence, μ may be taken equal to $\beta(r,z)F(\tau)$, where

$$\alpha_r\beta_r + \alpha_z\beta_z = \frac{\gamma+1}{2}\,\frac{X}{A^4}; \tag{35}$$

from (24), the left hand side may be replaced in terms of a derivative along a ray, to give

$$\frac{d\beta}{ds} = \frac{\gamma+1}{2}\,\frac{X}{A^3}. \tag{36}$$

To define β uniquely, it is stipulated that $\beta = 0$ on the initial wave front; then β is determined by integration along the rays. The solution (30),(31),(32) is now completed by the addition of the implicit relation for $\tau(t,r,z)$:

$$t = \alpha(r,z) - \tfrac{1}{2}(\gamma+1)\beta(r,z)F(\tau) + \tau. \tag{37}$$

In the problems of principal interest (in which the disturbance is caused by an explosion) the pressure and the sound speed increase at the head of the disturbance; thus, $F(\tau)$ is initially positive. It follows from (37) that the characteristic surfaces on which $F(\tau)$ is positive will eventually intersect the characteristic surfaces, $t - \alpha =$ constant, of the undisturbed region ahead where $F(\tau) = 0$. (Another way of expressing the same thing is that, in the (r,z) plane, wavelets carrying greater values of $F(\tau)$ travel faster and overtake the others). If this is allowed to occur, the solution is no longer single valued and ceases to have physical significance; the head shock must be fitted in this region in order that the characteristic surfaces meet it and are cut off by it before they intersect each other.

At any time, the following conditions must be satisfied across

the shock in the (r,z) plane:

(a) The component v_{tan} of the particle velocity tangential to the shock must be continuous.

(b) $v_n = 2(a-A)/(\gamma-1)$ where v_n is the component of the particle velocity normal to the shock, v_n and a being measured just behind the shock.

(c) $U_n = \frac{1}{2}(A+a+v_n)$ where U_n is the normal velocity of the shock.

These are deduced from the Rankine-Hugonist shock relations, neglecting terms of second and higher order in the strength of the shock. (b) and (c) are the obvious **extensions** of the two shock conditions which were used in [1] and (a) follows directly from the conservation of the tangential component of momentum. It may be noted that (c) expresses the fact that the shock velocity is the mean of the velocity of the wavelets on either side of it; in the $(t,\bar{r},z)$ space, it shows that the shock surface bisects the angle between the characteristic surfaces meeting it from either side.

It is clear that the shock may be assumed to be

$$t = a(r,z) - \sigma(r,z) \equiv \Sigma(r,z), \tag{38}$$

where σ/a is small, but again a crucial point is that σ becomes large compared to $t-a$. The component v_{tan} of the particle velocity is $Aua_z - Awa_r$ to the first order, and this must be zero by (a). But (30) and (31) show that the condition is already satisfied by our solution. The component v_n is $Aa_r u + Aa_z w$ to **first** order, and from (30),(31) this is $A(a_r^2+a_z^2)XF(\tau) = XA^{-1}F(\tau)$; hence (32) shows that (b) is also satisfied. The final condition (c) determines the shock. From (38),

$$U_n = \frac{1}{\sqrt{\Sigma_r^2+\Sigma_z^2}} \simeq A+A^3(a_r\sigma_r+a_z\sigma_z),$$

since σ/a is small; therefore, from (c), (32) and the above expression for v_n,

$$a_r\sigma_r+a_z\sigma_z = \frac{\gamma+1}{4}\frac{X}{A^4}F(\tau). \tag{39}$$

In addition, equating the values of $t-a$ given in (37) and (38), we have

(40) $$\sigma = \beta F(\tau) - \tau .$$

In principle, by eliminating τ, (39) and (40) determine σ as a function of r and z. It is more convenient, however, to find the relation between r, z, amd the value of τ at the shock; this relation specifies τ as a function of r, z and then (40) gives $\sigma(r,z)$. For this purpose, the right hand side of (39) is replaced from (35) by $\frac{1}{2}(\alpha_r\beta_r+\alpha_z\beta_z)F(\tau)$, and (40) is substituted for σ to give

(41) $$\tfrac{1}{2}(\alpha_r\beta_r+\alpha_z\beta_z)F(\tau) + (\beta F'(\tau)-1)(\alpha_r\tau_r+\alpha_z\tau_z) = 0;$$

On multiplying by $F(\tau)$, (41) can be written

(42) $$\left(\alpha_r \frac{\partial}{\partial r} + \alpha_z \frac{\partial}{\partial z}\right)\left\{\tfrac{1}{2}\beta F^2(\tau) - \int_0^{\tau} F(\tau')d\tau'\right\} = 0.$$

Since the operator is the derivative along the rays, the term in brackets is constant along each ray; but this quantity is zero on tho initial wave front, therefore

(43) $$\tfrac{1}{2}\beta(r,z)\ F^2(\tau) - \int_0^{\tau} F(\tau')d\tau' = 0,$$

at all points on the shock. It is of interest to observe that (43) is exactly the formula which was found for the shock in the problems involving only one space variable, except in those cases, of course, β was a function only of the one variable.

As already mentioned, (43) gives τ as a function of r and z at the shock; from it, $\sigma(r,z)$ may be obtained by (40) and the values of u,w and a just behind the shock by (27), (28) and (29). The strength of the shock is found similarly from

(44) $$\frac{p-P}{P} = \frac{R(z)\phi t}{P(z)} = \frac{\gamma X F(\tau)}{A^2} .$$

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